5.3 Inverse Functions

- Verify that one function is the inverse function of another function.
- Determine whether a function has an inverse function.
- Find the derivative of an inverse function.

Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

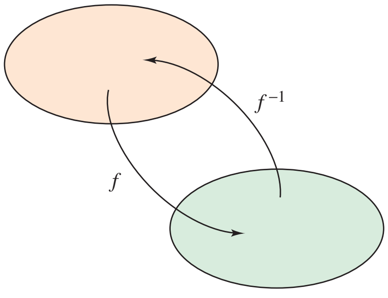
$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 5.10. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 5.10

REMARK Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of -1 as a superscript. That is, in general,

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Exploration

Finding Inverse Functions Explain how to "undo" each of the functions below. Then use your explanation to write the inverse function of f.

a. $f(x) = x - 5$

b. $f(x) = 6x$

c. $f(x) = \frac{x}{2}$

d. $f(x) = 3x + 2$

e. $f(x) = x^3$

f. $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$f(g(x)) = x$ for each x in the domain of g

and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but when it does, the inverse function is unique (see Exercise 96).

You can think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. So,

$$f(x) = x + c \quad \text{and} \quad f^{-1}(x) = x - c$$ Subtraction can be used to undo addition.

are inverse functions of each other and

$$f(x) = cx \quad \text{and} \quad f^{-1}(x) = \frac{x}{c}, \quad c \neq 0$$ Division can be used to undo multiplication.

are inverse functions of each other.

EXAMPLE 1 **Verifying Inverse Functions**

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

REMARK In Example 1, try comparing the functions f and g verbally.

For f: First cube x, then multiply by 2, then subtract 1.

For g: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"?

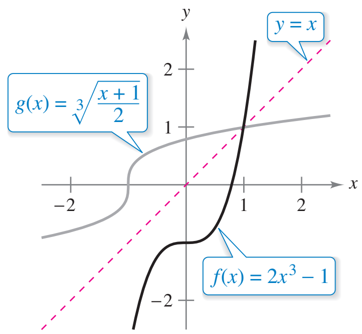
Solution Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x. The composition of f with g is

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$

The composition of g with f is

$$\begin{aligned} g(f(x)) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x. \end{aligned}$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 5.11).



f and g are inverse functions of each other.

Figure 5.11

In Figure 5.11, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the next theorem.

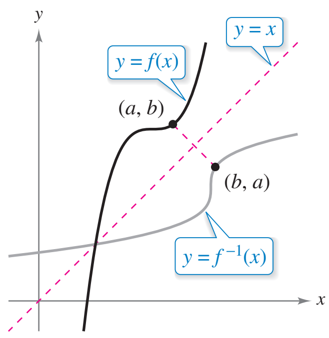
THEOREM 5.6 Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

Proof If (a, b) is on the graph of f, then $f(a) = b$, and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

So, (b, a) is on the graph of f^{-1}, as shown in Figure 5.12. A similar argument will prove the theorem in the other direction.

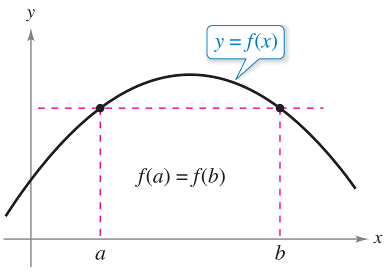


The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

Figure 5.12

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Existence of an Inverse Function



If a horizontal line intersects the graph of f twice, then f is not one-to-one.
Figure 5.13

Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 5.13). The next theorem formally states why the Horizontal Line Test is valid. (Recall from Section 3.3 that a function is *strictly monotonic* when it is either increasing on its entire domain or decreasing on its entire domain.)

THEOREM 5.7 The Existence of an Inverse Function

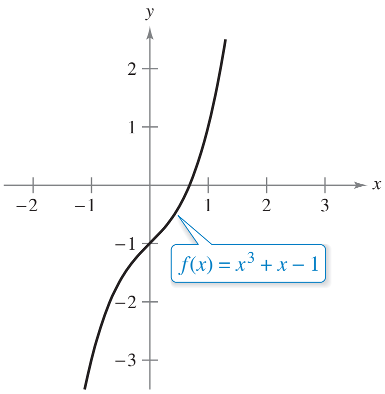
1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Proof The proof of the first part of the theorem is left as an exercise (See Exercise 97). To prove the second part of the theorem, recall from Section P.3 that f is one-to-one when for x_1 and x_2 in its domain

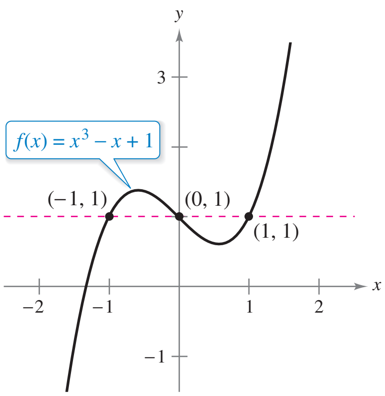
$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Now, choose x_1 and x_2 in the domain of f. If $x_1 \neq x_2$, then, because f is strictly monotonic, it follows that either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$. In either case, $f(x_1) \neq f(x_2)$. So, f is one-to-one on the interval.

See LarsonCalculus.com for Bruce Edwards's video of this proof.



(a) Because f is increasing over its entire domain, it has an inverse function.



(b) Because f is not one-to-one, it does not have an inverse function.
Figure 5.14

EXAMPLE 2 The Existence of an Inverse Function

a. From the graph of $f(x) = x^3 + x - 1$ shown in Figure 5.14(a), it appears that f is increasing over its entire domain. To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of x. So, f is strictly monotonic, and it must have an inverse function.

b. From the graph of $f(x) = x^3 - x + 1$ shown in Figure 5.14(b), you can see that the function does not pass the Horizontal Line Test. In other words, it is not one-to-one. For instance, f has the same value when $x = -1$, 0, and 1.

$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

So, by Theorem 5.7, f does not have an inverse function.

Often, it is easier to prove that a function *has* an inverse function than to find the inverse function. For instance, it would be difficult algebraically to find the inverse function of the function in Example 2(a).

GUIDELINES FOR FINDING AN INVERSE FUNCTION

1. Use Theorem 5.7 to determine whether the function $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

EXAMPLE 3 Finding an Inverse Function

Find the inverse function of $f(x) = \sqrt{2x - 3}$.

Solution From the graph of f in Figure 5.15, it appears that f is increasing over its entire domain, $[3/2, \infty)$. To verify this, note that

$$f'(x) = \frac{1}{\sqrt{2x-3}}$$

is positive on the domain of f. So, f is strictly monotonic, and it must have an inverse function. To find an equation for the inverse function, let $y = f(x)$, and solve for x in terms of y.

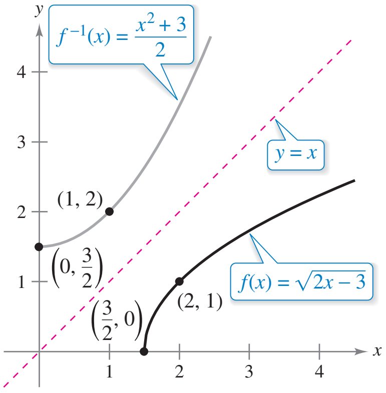
$$\sqrt{2x-3} = y \qquad \text{Let } y = f(x).$$

$$2x - 3 = y^2 \qquad \text{Square each side.}$$

$$x = \frac{y^2+3}{2} \qquad \text{Solve for } x.$$

$$y = \frac{x^2+3}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$f^{-1}(x) = \frac{x^2+3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$



The domain of f^{-1}, $[0, \infty)$, is the range of f.
Figure 5.15

The domain of f^{-1} is the range of f, which is $[0, \infty)$. You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x-3}\right)^2 + 3}{2} = \frac{2x - 3 + 3}{2} = x, \quad x \geq \frac{3}{2}$$

Theorem 5.7 is useful in the next type of problem. You are given a function that is *not* one-to-one on its domain. By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain.

EXAMPLE 4 Testing Whether a Function Is One-to-One

See LarsonCalculus.com for an interactive version of this type of example.

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real number line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, on which f is strictly monotonic.

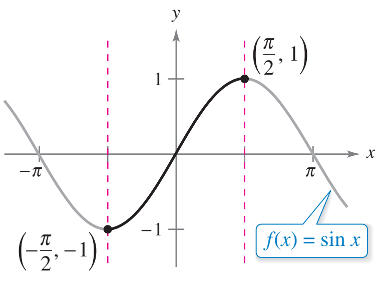
Solution It is clear that f is not one-to-one, because many different x-values yield the same y-value. For instance,

$$\sin(0) = 0 = \sin(\pi).$$

Moreover, f is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

$$f'(x) = \cos x$$

is positive there. Finally, because the left and right endpoints correspond to relative extrema of the sine function, you can conclude that f is increasing on the closed interval $[-\pi/2, \pi/2]$ *and* that on any larger interval the function is not strictly monotonic (see Figure 5.16).



f is one-to-one on the interval $[-\pi/2, \pi/2]$.
Figure 5.16

Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 5.8 follows from the reflective property of inverse functions, as shown in Figure 5.12.

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Exploration

Graph the inverse functions $f(x) = x^3$ and $g(x) = x^{1/3}$. Calculate the slopes of f at $(1, 1)$, $(2, 8)$, and $(3, 27)$, and the slopes of g at $(1, 1)$, $(8, 2)$, and $(27, 3)$. What do you observe? What happens at $(0, 0)$?

THEOREM 5.9 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 5 Evaluating the Derivative of an Inverse Function

Let $f(x) = \frac{1}{4}x^3 + x - 1$. (a) What is the value of $f^{-1}(x)$ when $x = 3$? (b) What is the value of $(f^{-1})'(x)$ when $x = 3$?

Solution Notice that f is one-to-one and therefore has an inverse function.

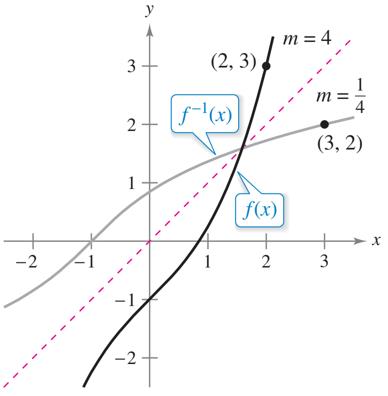
a. Because $f(x) = 3$ when $x = 2$, you know that $f^{-1}(3) = 2$.

b. Because the function f is differentiable and has an inverse function, you can apply Theorem 5.9 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

Moreover, using $f'(x) = \frac{3}{4}x^2 + 1$, you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$



The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).
Figure 5.17

In Example 5, note that at the point (2, 3), the slope of the graph of f is 4, and at the point (3, 2), the slope of the graph of f^{-1} is

$$m = \frac{1}{4}$$

as shown in Figure 5.17. In general, if $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \dfrac{dx}{dy}$. It follows from Theorem 5.9 that

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}.$$

This reciprocal relationship is sometimes written as

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

EXAMPLE 6 Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$), and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. (2, 4) and (4, 2) **b.** (3, 9) and (9, 3)

Solution The derivatives of f and f^{-1} are

$$f'(x) = 2x \quad \text{and} \quad (f^{-1})'(x) = \frac{1}{2\sqrt{x}}.$$

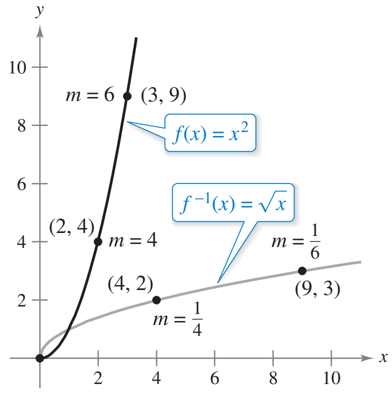
a. At (2, 4), the slope of the graph of f is $f'(2) = 2(2) = 4$. At (4, 2), the slope of the graph of f^{-1} is

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}.$$

b. At (3, 9), the slope of the graph of f is $f'(3) = 2(3) = 6$. At (9, 3), the slope of the graph of f^{-1} is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

So, in both cases, the slopes are reciprocals, as shown in Figure 5.18.



At (0, 0), the derivative of f is 0, and the derivative of f^{-1} does not exist.
Figure 5.18

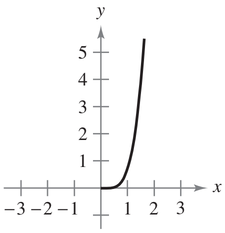
5.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Verifying Inverse Functions In Exercises 1–8, show that f and g are inverse functions (a) analytically and (b) graphically.

1. $f(x) = 5x + 1, \quad g(x) = \dfrac{x-1}{5}$
2. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3-x}{4}$
3. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$
4. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1-x}$
5. $f(x) = \sqrt{x-4}, \quad g(x) = x^2 + 4, \quad x \ge 0$
6. $f(x) = 16 - x^2, \quad x \ge 0, \quad g(x) = \sqrt{16-x}$
7. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$
8. $f(x) = \dfrac{1}{1+x}, \quad x \ge 0, \quad g(x) = \dfrac{1-x}{x}, \quad 0 < x \le 1$

Matching In Exercises 9–12, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

(a)


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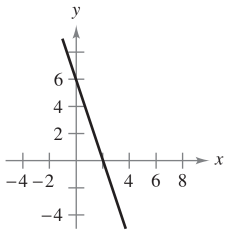
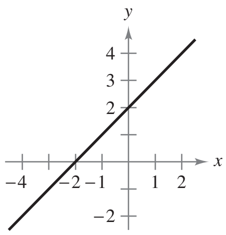
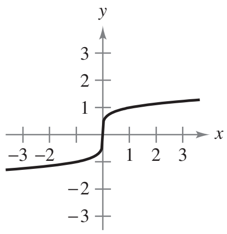


(c)


(d)


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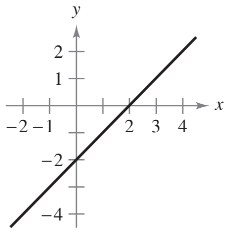

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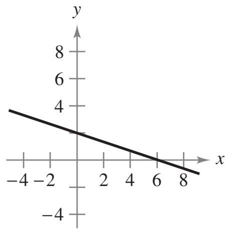

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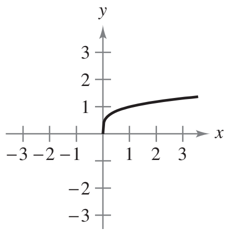

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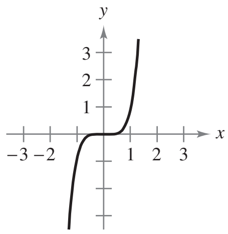


Using the Horizontal Line Test In Exercises 13–22, use a graphing utility to graph the function. Then use the Horizontal Line Test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function.

13. $f(x) = \frac{3}{4}x + 6$
14. $f(x) = 5x - 3$
15. $f(\theta) = \sin\theta$
16. $f(x) = \dfrac{6x}{x^2+4}$
17. $h(s) = \dfrac{1}{s-2} - 3$
18. $g(t) = \dfrac{1}{\sqrt{t^2+1}}$
19. $f(x) = \ln x$
20. $f(x) = 5x\sqrt{x-1}$
21. $g(x) = (x+5)^3$
22. $h(x) = |x+4| - |x-4|$

Determining Whether a Function Has an Inverse Function In Exercises 23–28, use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse function.

23. $f(x) = 2 - x - x^3$
24. $f(x) = x^3 - 6x^2 + 12x$
25. $f(x) = \dfrac{x^4}{4} - 2x^2$
26. $f(x) = x^5 + 2x^3$
27. $f(x) = \ln(x-3)$
28. $f(x) = \cos\dfrac{3x}{2}$

Verifying a Function Has an Inverse Function In Exercises 29–34, show that f is strictly monotonic on the given interval and therefore has an inverse function on that interval.

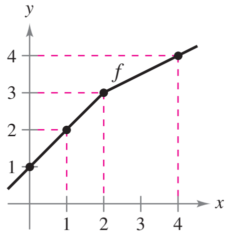
29. $f(x) = (x-4)^2, \quad [4, \infty)$
30. $f(x) = |x+2|, \quad [-2, \infty)$
31. $f(x) = \dfrac{4}{x^2}, \quad (0, \infty)$
32. $f(x) = \cot x, \quad (0, \pi)$
33. $f(x) = \cos x, \quad [0, \pi]$
34. $f(x) = \sec x, \quad \left[0, \dfrac{\pi}{2}\right)$

Finding an Inverse Function In Exercises 35–46, (a) find the inverse function of f, (b) graph f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs, and (d) state the domain and range of f and f^{-1}.

35. $f(x) = 2x - 3$
36. $f(x) = 7 - 4x$
37. $f(x) = x^5$
38. $f(x) = x^3 - 1$
39. $f(x) = \sqrt{x}$
40. $f(x) = x^2, \quad x \ge 0$
41. $f(x) = \sqrt{4-x^2}, \quad 0 \le x \le 2$
42. $f(x) = \sqrt{x^2-4}, \quad x \ge 2$
43. $f(x) = \sqrt[3]{x-1}$
44. $f(x) = x^{2/3}, \quad x \ge 0$
45. $f(x) = \dfrac{x}{\sqrt{x^2+7}}$
46. $f(x) = \dfrac{x+2}{x}$

Finding an Inverse Function **In Exercises 47 and 48, use the graph of the function f to make a table of values for the given points. Then make a second table that can be used to find f^{-1}, and sketch the graph of f^{-1}. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

47.



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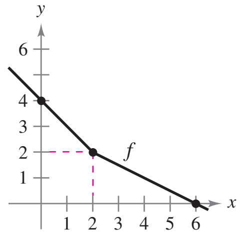


49. Cost You need 50 pounds of two commodities costing \$1.25 and \$1.60 per pound.

(a) Verify that the total cost is $y = 1.25x + 1.60(50 - x)$, where x is the number of pounds of the less expensive commodity.

(b) Find the inverse function of the cost function. What does each variable represent in the inverse function?

(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.

(d) Determine the number of pounds of the less expensive commodity purchased when the total cost is \$73.

50. Temperature The formula $C = \frac{5}{9}(F - 32)$, where $F \geq -459.6$, represents Celsius temperature C as a function of Fahrenheit temperature F.

(a) Find the inverse function of C.

(b) What does the inverse function represent?

(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.

(d) The temperature is 22°C. What is the corresponding temperature in degrees Fahrenheit?

Testing Whether a Function Is One-to-One **In Exercises 51–54, determine whether the function is one-to-one. If it is, find its inverse function.**

51. $f(x) = \sqrt{x - 2}$

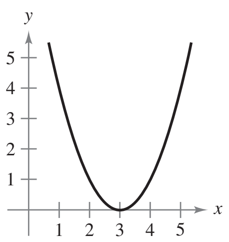
52. $f(x) = -3$

53. $f(x) = |x - 2|, \quad x \leq 2$

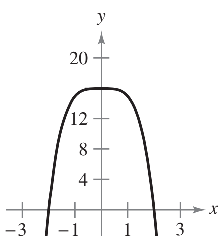
54. $f(x) = ax + b, \quad a \neq 0$

Making a Function One-to-One **In Exercises 55–58, delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function. (*Note:* There is more than one correct answer.)**

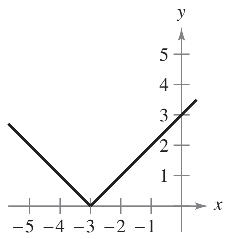
55. $f(x) = (x - 3)^2$



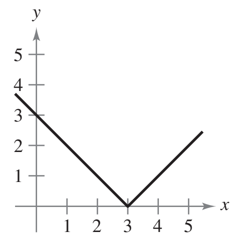
56. $f(x) = 16 - x^4$



57. $f(x) = |x + 3|$



58. $f(x) = |x - 3|$



Think About It **In Exercises 59–62, decide whether the function has an inverse function. If so, what is the inverse function?**

59. $g(t)$ is the volume of water that has passed through a water line t minutes after a control valve is opened.

60. $h(t)$ is the height of the tide t hours after midnight, where $0 \leq t < 24$.

61. $C(t)$ is the cost of a long distance call lasting t minutes.

62. $A(r)$ is the area of a circle of radius r.

Evaluating the Derivative of an Inverse Function **In Exercises 63–70, verify that f has an inverse. Then use the function f and the given real number a to find $(f^{-1})'(a)$. (*Hint:* See Example 5.)**

63. $f(x) = 5 - 2x^3, \quad a = 7$

64. $f(x) = x^3 + 2x - 1, \quad a = 2$

65. $f(x) = \frac{1}{27}(x^5 + 2x^3), \quad a = -11$

66. $f(x) = \sqrt{x - 4}, \quad a = 2$

67. $f(x) = \sin x, \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}, \quad a = \dfrac{1}{2}$

68. $f(x) = \cos 2x, \quad 0 \leq x \leq \dfrac{\pi}{2}, \quad a = 1$

69. $f(x) = \dfrac{x + 6}{x - 2}, \quad x > 2, \quad a = 3$

70. $f(x) = \dfrac{x + 3}{x + 1}, \quad x > -1, \quad a = 2$

Using Inverse Functions **In Exercises 71–74, (a) find the domains of f and f^{-1}, (b) find the ranges of f and f^{-1}, (c) graph f and f^{-1}, and (d) show that the slopes of the graphs of f and f^{-1} are reciprocals at the given points.**

	Functions	Point
71.	$f(x) = x^3$	$\left(\frac{1}{2}, \frac{1}{8}\right)$
	$f^{-1}(x) = \sqrt[3]{x}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
72.	$f(x) = 3 - 4x$	$(1, -1)$
	$f^{-1}(x) = \dfrac{3 - x}{4}$	$(-1, 1)$
73.	$f(x) = \sqrt{x - 4}$	$(5, 1)$
	$f^{-1}(x) = x^2 + 4, \quad x \geq 0$	$(1, 5)$
74.	$f(x) = \dfrac{4}{1 + x^2}, \quad x \geq 0$	$(1, 2)$
	$f^{-1}(x) = \sqrt{\dfrac{4 - x}{x}}$	$(2, 1)$

Using Composite and Inverse Functions **In Exercises 75–78, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the given value.**

75. $(f^{-1} \circ g^{-1})(1)$

76. $(g^{-1} \circ f^{-1})(-3)$

77. $(f^{-1} \circ f^{-1})(6)$

78. $(g^{-1} \circ g^{-1})(-4)$

Using Composite and Inverse Functions **In Exercises 79–82, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the given function.**

79. $g^{-1} \circ f^{-1}$

80. $f^{-1} \circ g^{-1}$

81. $(f \circ g)^{-1}$

82. $(g \circ f)^{-1}$

WRITING ABOUT CONCEPTS

83. In Your Own Words Describe how to find the inverse function of a one-to-one function given by an equation in x and y. Give an example.

84. A Function and Its Inverse Describe the relationship between the graph of a function and the graph of its inverse function.

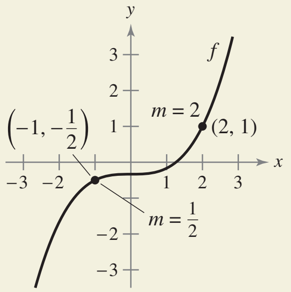
Explaining Why a Function Is Not One-to-One **In Exercises 85 and 86, the derivative of the function has the same sign for all x in its domain, but the function is not one-to-one. Explain.**

85. $f(x) = \tan x$

86. $f(x) = \dfrac{x}{x^2 - 4}$

87. Think About It The function $f(x) = k(2 - x - x^3)$ is one-to-one and $f^{-1}(3) = -2$. Find k.

88. HOW DO YOU SEE IT? Use the information in the graph of f below.



(a) What is the slope of the tangent line to the graph of f^{-1} at the point $\left(-\frac{1}{2}, -1\right)$? Explain.

(b) What is the slope of the tangent line to the graph of f^{-1} at the point $(1, 2)$? Explain.

True or False? **In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. If f is an even function, then f^{-1} exists.

90. If the inverse function of f exists, then the y-intercept of f is an x-intercept of f^{-1}.

91. If $f(x) = x^n$, where n is odd, then f^{-1} exists.

92. There exists no function f such that $f = f^{-1}$.

93. Making a Function One-to-One

(a) Show that $f(x) = 2x^3 + 3x^2 - 36x$ is not one-to-one on $(-\infty, \infty)$.

(b) Determine the greatest value c such that f is one-to-one on $(-c, c)$.

94. Proof Let f and g be one-to-one functions. Prove that

(a) $f \circ g$ is one-to-one.

(b) $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

95. Proof Prove that if f has an inverse function, then $(f^{-1})^{-1} = f$.

96. Proof Prove that if a function has an inverse function, then the inverse function is unique.

97. Proof Prove that a function has an inverse function if and only if it is one-to-one.

98. Using Theorem 5.7 Is the converse of the second part of Theorem 5.7 true? That is, if a function is one-to-one (and therefore has an inverse function), then must the function be strictly monotonic? If so, prove it. If not, give a counterexample.

99. Concavity Let f be twice-differentiable and one-to-one on an open interval I. Show that its inverse function g satisfies

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

When f is increasing and concave downward, what is the concavity of $f^{-1} = g$?

100. Derivative of an Inverse Function Let

$$f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}.$$

Find $(f^{-1})'(0)$.

101. Derivative of an Inverse Function Show that

$$f(x) = \int_2^x \sqrt{1 + t^2}\, dt$$

is one-to-one and find

$(f^{-1})'(0)$.

102. Inverse Function Let

$$y = \frac{x - 2}{x - 1}.$$

Show that y is its own inverse function. What can you conclude about the graph of f? Explain.

103. Using a Function Let $f(x) = \dfrac{ax + b}{cx + d}$.

(a) Show that f is one-to-one if and only if $bc - ad \neq 0$.

(b) Given $bc - ad \neq 0$, find f^{-1}.

(c) Determine the values of a, b, c, and d such that $f = f^{-1}$.